Name:

ASTR 3710: Midterm #2

Open book, no internet or phone-a-friend but notes of any kind are OK. Remember that it's easier to give partial credit for answers that are concise and legible!

Possibly useful formulae and constants

Hill radius $R_{Hill} = \left(\frac{M_P}{3M_*}\right)^{1/3} a$

Energy of Keplerian orbits $E = -\frac{GM}{2a}$

Pericenter / apocenter r_p = a (1-e), r_a = a (1+e)

Statistical model for planetary growth $\frac{dM}{dt} = \frac{1}{2}\Sigma_p \Omega \pi R_s^2 \left(1 + \frac{v_{esc}^2}{\sigma^2}\right)$

1 AU = 1.496×10^{13} cm

Solar mass = 1.989×10^{33} g

Jupiter mass = 1.9×10^{30} g

G = 6.67×10^{-8} cm^3 g^{-1} s^{-2}

(1) How would you expect the Solar System to be different if the Solar nebula had been dispersed on a time scale of less than a million years, instead of the 5-10 million years that we think was actually the case?

The terrestrial planet formation time scale is much greater than even 10 Myr, so their properties would likely have remained similar. The gas giant take several million years to reach "runaway" envelope accretion, so a gas disk time scale of less than a Myr would not have given them enough time to reach this stage. Therefore, the likely outcome would have been 4 ice giants rather than 2 gas giants and 2 ice giants.

(2) Consider a hypothetical protoplanetary disk of the same size as the Solar nebula, but containing only 1/10 the mass of solid material. How would you expect the terrestrial planets formed from such a disk to differ from those in the Solar System?

Based on the stability considerations we discussed, one would expect more terrestrial planets whose masses would be lower than the Solar System by *more than* a factor of ten.

(3) Suppose that the Sun were to "instantaneously" (on a time scale << 1 year) blow off a spherical shell of gas containing 10% of its mass. Considering *only* the gravitational effects of this loss, how would Earth's orbit be affected?

One can work this out quantitatively, but the important physics is the critical thing. Recall that a spherically symmetric mass distribution produces the same gravitational field as a point mass, and that there is zero force inside a spherical shell of mass. Hence, the shell causes no change to the gravitational force felt by the Earth while it is interior to the Earth's orbit. Once it passes by the Earth, the effect is that the gravity of the Sun is instantaneously reduced by 10%.

This means that after the shell has swept past the Earth, the Earth has *too large* a velocity for a circular orbit at 1 AU. Hence, the semi-major a increases. However, there is no change to the angular momentum, as there are no non-radial forces involved. Hence, the eccentricity also changes, becoming non-zero.

(4) A future observation shows that the frequency of hot Jupiters around stars less than 100 Myr old is only half as large as the frequency around Gyr old stars. However, *all* of the hot Jupiters around young stars orbit in the equatorial plane of their host stars. What would this observation imply for theories for how hot Jupiters formed?

The mechanism that is expected to produce aligned hot Jupiters is gas disk migration, which also occurs early (while the gas is still around). So the simplest (not the only) explanation of the hypothetical observations would be:

a) About half of the hot Jupiters form from gas disk migration. This happens quickly, so they are present around even young stars, and results in orbits that line up with the stellar equator.
b) The other half form from dynamical processes (e.g. secular chaos, or Kozai resonance in binaries) which are slower (taking more than 100 Myr) and lead to misaligned orbits.

(5) Numerical calculations show that a 3 planet system made up of 3 Jupiter mass planets on circular orbits at 5, 8 and 13 AU is likely to be stable for the lifetime of a Solar mass star, but that packing the planets significantly more closely leads to instability.

(a) Consider a Kepler planetary system that also contains 3 Jupiter mass planets, but where the innermost planet is at 0.1 AU. How close could the second and third planets in this system be for it to be marginally stable?

Recall that stability depends on separation measured in units of the Hill radius, which is just a linear function of semi-major axis a. So a planetary system in which *all* the orbits are scaled down by a constant factor is about as stable as the original system (more accurately, ignoring any tidal or relativistic effects, it is exactly as

stable over the *same number of orbits*). So scaling the given system down so that the inner planet is at 0.1 AU, one finds a_2 = 0.16 AU, a_3 = 0.26 AU.

(b) Suppose that the three planets all have masses of 0.3 Jupiter masses. If the innermost planet remains at 5 AU, how close could the other planets be to maintain marginal stability?

The Hill radius scales with planet mass as $M^{1/3}$. So, roughly, we can reduce the separations by $0.3^{1/3} = 0.67$. So a_2 = 7 AU, a_3 = 10.4 AU. (This isn't quite right, but will do for the purposes of this question.)

(6) A spacecraft has a circular orbit about the Sun at 1 AU. Is it easier (in the sense of requiring less energy) to place the spacecraft into an orbit that will escape the Solar System, or to place it into an orbit that will cause it to plunge into the Sun? Explain your answer as quantitatively as possible.

Energy of a circular orbit (per unit mass): $E = -GM / 2a$

To escape, need $\Delta E = + GM / 2a$

To hit Sun, need to reduce azimuthal velocity to zero, requiring energy $(1/2)\, v_K^2 = GM / 2a$

Hence, the two options are of comparable difficulty

(7) Colorado's *MAVEN* spacecraft, due for launch next week, will reach Mars via an orbit whose pericenter is at 1 AU, and whose apocenter is at 1.5 AU.

(a) Calculate the semi-major axis and eccentricity of this orbit.

$r_p = a\,(1-e)$
$r_a = a\,(1+e)$

Adding $r_p + r_a = 2a$

Hence, a = 1.25 AU, e = 0.2

(b) How long in Earth years will it take to reach Mars?

It should take ½ an orbit (if Earth is at pericenter, and Mars at apocenter). Since $P^2 = a^3$ (with P in years, and a in AU) at 1.25 AU $(P/2) = (1/2) \times a^{3/2} = 0.7$ years.

(8) Consider the simple model of planetary growth in the limit where gravitational focusing is negligible. Find an expression for M / dM/dt and use

this to say whether larger protoplanets grow faster or slower than smaller ones in this regime.

Ignoring gravitational focusing, $dM/dt = (1/2)\Sigma_p\Omega\pi R_s^2$

Since mass $M \sim R_s^3$ we have $R_s^2 \sim M^{2/3}$

Hence, $M / dM/dt \sim M^{1/3}$ – larger protoplanets have a longer doubling time scale and grow slower

(9) Suppose that Jupiter, in a circular orbit at 5.2 AU, ejects planetesimals from the Solar System. The planetesimals can be assumed to start on circular orbits near that of Jupiter, and to receive a kick from Jupiter that only just causes them to escape the Sun's gravity.

(a) Does Jupiter move inward or outward as a result of ejecting the planetesimals?

Inward

(b) If Jupiter's orbit remains circular, estimate the mass of planetesimals that need to be ejected to result in Jupiter's semi-major axis changing by 1 AU.

Let Jupiter's initial orbit be a_i and the final orbit be a_f

Change in energy for Jupiter is $\Delta E = -GMM_J / 2 \ (1/a_f - 1/a_i)$

Th energy per unit mass needed to eject planetesimals is $\varepsilon = GM / 2a_p$

Mass required is $m = \Delta E / \varepsilon = M_J \, a_p \ (1/a_f - 1/a_i)$

With $a_i = 5.2$ AU, $a_f = 4.2$ AU, and approximating $a_p = 4.7$ AU one gets a mass of about $0.2 \ M_J$